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A TRANSFER MAP FOR SINGULAR FIBRATIONS

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ABSTRACT
Using a new group of singular chains on polyhedra, a transfer map is defined on the chain level for fibrations with singularities. These include usual transfer on homology as well as Oliver transfer and other cases.

In the present note I would like to demonstrate how the ideas of Smith, Munkholm and Oliver ([S], [Mu], [O]) can be combined to yield an elementary definition of a transfer map on homology with a ring of coefficient R, for certain PL-maps that I call here singular fibrations. A singular fibration is a PL-map $E \to B$ that is a fibre map away from a "small" subcomplex $W \subset B$. This transfer in addition to its conceptual simplicity enjoys the following properties:

(1) It is defined for a class of PL-map that includes, by virtue of results about triangulation ([I], [V]), the class $X/H \to X/G$ of Oliver, for $H \subset G$ and X a G-space.

(2) It is natural with respect to composition of fibre maps.

(3) It behaves well with respect to any decomposition of total space into a union of two singular fibrations along a third one: If $E = E_1 \cup E_2$ with $E_3 = E_1 \cap E_2$ where E, E_i ($i = 1, 2, 3$) are all fibrations with small singularities, then the corresponding transfers add up correctly, $t = t_1 + t_2 - t_3$.

Unfortunately, in order to define this chain level transfer in a natural way one has to blow up the usual singular chains to a much bigger group that can be called *compact singular chains* or compact chains for short. A typical compact

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singular chain in X is represented by a PL-map $\Delta_q \times L \to X$ where L and X are polyhedra.

0.1. THEOREM. *Let $\pi : M^m \to N^n$ be a simplicial map of compact manifolds with N^n connected that is a Serre fibre map away from a codimension two subcomplex $W \subset N^n$. Then π admits a (co)-homology transfer map $\tau_* : H_*(N^n) \to H_*(M^m)$ with $\pi_* \circ \tau_* = \chi(F) \cdot 1$, where F is a fibre away from W.*

In some way this is just a sample result. The method could be used to give transfer in other situations (see, e.g., Section 3 below).

Let us illustrate the use of compact chains as follows: A typical element in the compact chain complex of X is represented by a map $\Delta_q \times K \to X$ where K can be any polyhedron, Δ_q is the standard q-simplex and faces are gotten by restriction to $\Delta_{q-1} \times K$. Let $X \to S^1$ be a fibre bundle with fibre F. Then there is an obvious 'unfolding' map $u : \Delta_1 \times F \to X$ over the gluing map $\Delta_1 \to S^1$, presenting X as a mapping torus for some PL-homeomorphism $F \to F$. It turns out that, in the framework of compact chains, u is a cycle that can be taken as the image under transfer of the canonical cycle $\Delta_1 \to S^1$ in $C_1(S^1)$.

0.2. DEFINITION AND PROPOSITION. *Denote by $\bar{C}_q(X)$ the free abelian group generated by continuous maps $\Delta_q \times \Delta_n \to X$ subject to certain collapsing relations* (1.1) *below, where $0 \leqq n < \infty$ and faces come from the Δ_q-factor. The natural inclusion map of the singular chain complex $S_* \subseteq \bar{C}_*$ is a chain homotopy equivalence.*

Using this easily proved proposition we interpret any map $\Delta_q \times K \to X$ for a triangulated polyhedron K as a q-chain and then for a non-triangulated one and define transfer using these chains.

0.3. *Compact chains.* In order to interpret any PL-map of the form $\Delta_q \times K \to X$ in the PL-category as a chain, we define in subsection 1.4 compact chains as a quotient group of the group $\bar{C}_q(X)$. In the second section compact chains are used to construct transfer maps. It should be noted that Munkholm has defined transfer map for PL-fibrations on the chain level [Mu], and we have used his ideas in the present note. In some sense, we show here how to use Munkholm's chain level map to enhance Smith's construction to singular fibrations with non-discrete fibres.

I would like to thank Professors L. Smith, R. Oliver and S. Weinberger for illuminating discussion on transfer.

1. Compact chain complexes

Consider the chain complex $\bar{C}_*$ from 0.2 above. The set of generators for $\bar{C}_q(X)$ are maps $\alpha : \Delta_q \times \Delta_n \to X$ subject to the following collapsing relations. For any order preserving simplicial collapse map (degeneracy) $S_i : \Delta_{n+1} \to \Delta_n$ given on barycentric coordinates by

$$(1.1) \qquad S_i : (t_0, t_1 \dots t_i \dots, t_{n+1}) \mapsto (t_0, \dots t_i + t_{i+1}, \dots, \dots t_{n+1})$$

the map α above is identified with the composite $\alpha \circ (1 \times S_i)$:

$$\Delta_q \times \Delta_{n+1} \to \Delta_q \times \Delta_n \to X.$$

The free abelian group $\bar{C}_q(X)$ on the equivalence classes $[\alpha] = [\alpha \cdot S_i]$ is called the group of *thick q-chains*. The i-th face of a generator $[\alpha]$ as above is given by $[\alpha \cdot d_i]$ where

$$d_i : \Delta_{q-1} \times \Delta_n \xrightarrow{d_i \times 1} \Delta_q \times \Delta_n$$

since d_i and s_i operating on different factors of $\Delta_q \times \Delta_n$ is well defined on equivalent classes.

There is an obvious inclusion map $j_* : S_*(X) \to \bar{C}_*(X)$ from the singular chain complex $S_*(X)$ to the thick chains given by $j_q : (\Delta_q \to X) \mapsto \Delta_q \times \Delta_0 \to X$.

1.2. PROPOSITION. *The inclusion j_* is a chain homotopy equivalence.*

PROOF. To define a chain homotopy inverse for j_* let $\varphi_q : \bar{C}_q(X) \to S_q(X)$ be the "evaluation at the first vertex map" sending the class $[\alpha]$ represented by $\alpha : \Delta_q \times \Delta_n \to X$ to the restriction of α to $\Delta_q \times \Delta_0$. Namely $\varphi_q(\chi) = \alpha(\chi, (1, 0 \cdots 0))$, where $S \in \Delta_q$. The value of $\varphi_q \times (\alpha)$ certainly does not depend on the representative $\alpha \in [\alpha]$, as can be seen by inspecting the equivalence relation given in (1.1) above. Now clearly φ_* is a chain map and $\varphi_* \circ j_* = 1_{s_*}$. To construct a homotopy $\bar{H}_q$ between $j_* \circ \varphi_*$ and $1_{\bar{C}_*}$ let H_q be the homotopy in Δ_n that shrinks 1_{Δ_n} to the collapse $\Delta_n \to \Delta_0 \to \Delta_n$. Thus $H : I \times \Delta_n \to \Delta_n$ with $H(t, x) = (t, 0 \cdots 0) + (1 - t)\chi$ in barycentric coordinates, $x \in \Delta_n$, $0 \leqq t \leqq 1$. Given an equivalence class $[\alpha] \in \bar{C}_q(X)$ with representative $\alpha : \Delta_q \times \Delta_n \to X$, define $h_\alpha : \Delta_q \times \Delta_1 \times \Delta_n \to X$ to be $H_\alpha(y, t, x) = \alpha(y, H(t, x))$. Now let ΣZ_i^{q+1} be the canonical chain in $S_{q+1}(\Delta_q \times \Delta_1)$ gotten by the standard triangulation of that prism: $Z_i : \Delta_{q+1} \to \Delta_q \times \Delta_1$. Then we can take the value of the chain homotopy $\bar{H}_q : \bar{C}_q(X) \to C_{q+1}(X)$ by

$$(1.3) \qquad \bar{H}_q(\alpha) = \sum_i H_\alpha \circ (Z_i \times 1)$$

where $Z_i \times 1$ is the map

$$\Delta_{q+1} \times \Delta_n \to \Delta_q \times \Delta_1 \times \Delta_n.$$

It is a straightforward verification that the value $\bar{H}_q(\alpha)$ depends only on the equivalence class $[\alpha]$ of α and thus that $\bar{H}_*$ is a well-defined chain homotopy as needed. □

1.4. *Compact chains.* Given a map $f: \Delta_q \times K \to X$ where K is a triangulated polyhedron, one can immediately associate with it a well-defined element in $\bar{C}_q(X)$, namely

$$|f| = \sum_{\sigma \in K'} (-1)^n \left[\Delta_q \times \Delta_n \xrightarrow{f_\sigma} X\right]; \qquad n = \dim \sigma$$

where f_σ is the restriction of f to the space $\Delta_q \times \Delta_n$ and where $\sigma: \Delta_n \subset K'$ is an ordered n-simplex in the standard barycentric subdivision K' of K. Our main interests lie in chains gotten as $[f]$ above, modulo a certain equivalence relation (1.7). We consider next the category CPL(2) of compact polyhedra pairs and PL-maps between them (see [R–S]). Using the usual procedures it is immediate to check the following:

1.5. PROPOSITION. *The category* CPL(2) *of compact polyhedral pairs is an admissible category in the sense of* [E–S].

PROOF. Straightforward.

1.6. REMARK. We now proceed to define another chain complex functor on CPL(2), whose homology can be easily seen to satisfy the Eilenberg–Steenrod axiom for a homology functor on an admissible category of pairs. These chains are necessary in order to render the *transfer map* defined in Section 2 below natural on the level of chains.

These chain groups are certain quotient groups of $\bar{C}_*(X)$ in which two different triangulations of the same PL map $\Delta_q \times K \to X$ will yield the same element. The quotient group will be called the group of compact chains of X and will be denoted by $CC_*(X)$. The group $CC_*(X)$ is a group of equivalence classes in $\bar{C}_*(X)$ under 'Euler equivalence' defined as follows:

1.7. DEFINITION. Two 0-chains Σf_i, Σg_j in $\bar{C}_0(X)$ where $f_i, g_i: \Delta_0 \times \Delta_n \to X$ are Euler equivalent if for any point $x \in X$ one has the equality

$$\sum_i \chi(f_i^{-1}(x)) = \sum_j \chi(g_j^{-1}(x))$$

(in words, if they give the same 'weight' to each point $x \in X$ via the Euler characteristic of the inverse image). Two q-chains in $\check{C}_q(X)$, namely Σf_i, Σg_j where $f_i, g_j : \Delta_q \times \Delta_n \to X$, are Euler equivalent if for any $t \in \Delta_q$ the corresponding 0-chains $\Sigma f_i(t, \text{—})$, $\Sigma g_j(t, \text{—})$ are Euler equivalent as 0-chains.

1.8. Remarks. (i) The above definition makes sense, since for any $x \in X$ the space $f_i^{-1}(x)$ is a compact polyhedron so that its Euler characteristic x is well defined.

(ii) Using a group map $\mathbf{Z} \to G$ one can define Euler equivalence with respect to a quotient group G.

1.9. Definition and Proposition. *The Euler equivalence relation respects the group law in $\check{C}_*(X)$ and therefore one can form the quotient group $CC_*(X) = \check{C}_*(X)/\sim$ called the group of compact chains.*

1.10. Proposition. *Let K', K'' denote two triangulation of K. Then any p.l. map $h : K \times \Delta_q \to X$ will yield via* 1.4 *Euler equivalent q-chains, so that any p.l. map h defines a unique element in $CC_q(X)$.*

Proof. Change of triangulation does not affect the Euler characteristics.

The main technical result of this section is

1.11. Theorem. *The natural quotient map $\check{C}_*(X) \to CC_*(X)$ induces an isomorphism on homology.*

Proof. It is not hard to check that the homology of $CC_*(X)$ satisfies all the Eilenberg–Steenrod axioms for integral homology. The proof parallels the usual proof for singular chains except one must remember that $CC_q(X)$ is not a free abelian group and modify the proof to take this into account. The only slight difficulty arises, in proving excision where one needs to consider compact chains dominated by a cover. For this, one takes a thick q-simplex $\Delta_n \times \Delta_q \to X$ and subdivides both Δ_n and Δ_q.

2. Transfer on compact chains

It turns out that for every simplicial map $\pi : K \to L$ there is a natural "pre-transfer" map $\bar{\tau}_q : C_q(L) \to CC_q(K)$ from the simplicial q-chains on L to the compact q-chains on K. However, this map is *not* in general a *chain map*.

If π is a PL-fibration then one can see that $\bar{\tau}_*$ is in fact a chain map. Moreover, in certain other cases one can modify $\bar{\tau}_*$ so as to turn it into a chain map. In particular we consider here maps that are Serre fibrations away from a subcomplex $W \subset L$ that is small enough in a well-defined sense. The motivating example is due to L. Smith who considered the case when all fibres are discrete [S, pp. 490–491].

2.1. Definition of $\bar{\tau}_n$. Notice that for any simplicial map $\pi : K \to \Delta_q$ to the standard q-simplex Δ_q there corresponds a canonical map $\tilde{\pi}$ over Δ_q:

One has a commutative square:

$$\begin{array}{ccc} \Delta_q \times K_b & \xrightarrow{\tilde{\pi}} & K \\ {\scriptstyle pr_1}\downarrow & & \downarrow{\scriptstyle \pi} \\ \Delta_q & = & \Delta_q \end{array}$$

such that:

(i) K_b is the inverse image of the barycenter $b \in \Delta_q$ with the induced simplicial structure.

(ii) $\tilde{\pi}$ is a relative homeomorphism.

(iii) Let $\Delta_{q-1} \subseteq \Delta_q$ be a face and let $\partial\pi : \partial K \to \Delta_{q-1}$ be the restriction of π to $\pi^{-1}(\Delta_{q-1}) \equiv \partial K_\alpha$. Let the map $\psi : K_b \to \partial K_\alpha$, where ∂K_α is $(\partial\pi)^{-1}$ (barycenter α of Δ_{q-1}) given by $\psi = \tilde{\pi} \mid \{\alpha\} \times K_b$. The following diagram is commutative:

$$\begin{array}{ccccccccccc} \Delta_{q-1} \times K_b & \xrightarrow{\text{res}} & \Delta_q \times K_b & \xrightarrow{\tilde{\pi}} & K & \leftarrow & \partial K & \xleftarrow{(\partial\tilde{\pi})} & \Delta_{q-1} \times \partial K_\alpha & \xleftarrow{\psi} & \Delta_{q-1} \times K_b \\ \downarrow & & \downarrow & & {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \partial\pi} & & \downarrow & & \downarrow \\ \Delta_{q-1} & \to & \Delta_q & = & \Delta_q & \leftarrow & \Delta_{q-1} & = & \Delta_{q-1} & = & \Delta_{q-1} \end{array} \tag{2.2}$$

The map ψ sends the barycenter $\hat{e}_b$ of $e_b \in K_b$ to the barycenter $\hat{e}_\alpha$ of $e_\alpha := e_b \cap \pi^{-1}(\Delta_{n-1})$, and therefore preserves order on the simplices.

Just as in [H, p. 105] it suffices to consider the case K is Δ_m with the canonical triangulation, π simplicial. Then these facts follow immediately from naturality with respect to face and collapse from the corresponding statement for $K = \Delta_m$, i.e. for simplicial map $\Delta_m \to \Delta_q$. These in turn are easy to check, e.g., by the formulae given in [Z].

Now for a simplicial map $\pi : K \to L$ of simplicial complexes with $\pi^{-1}(x)$ a compact polyhedron for each $x \in L$ one can define a *pre-transfer* map $\bar{\tau} : C_q(L) \to CC_q(K)$ that is *not*, in general, a chain map. $\bar{\tau}_q$ is defined by naturality via its value for a map $\pi : K \to \Delta_q$. In the last case one defines $\bar{\tau}(\iota_q)$, where ι_q is the identity simplex $\Delta_q \subseteq \Delta_q$ in $C_q(\Delta_q)$, to be the class in $C_q(L)$ of the map $\tilde{\pi}_q : \Delta_q \times K_b \to K$ over Δ_q. For a general $\pi : K \to L$ one takes the value of $\bar{\tau}_q$ on a q-simplex $\sigma \in L$ to be the composition

$$\Delta_q \times K_b \xrightarrow{\tilde{\pi}} K_\sigma \to K \tag{2.3}$$

where $K_\sigma = \pi^{-1}(\sigma)$, $\pi_\sigma = \pi \mid K_\sigma$, and b is the barycenter of σ. One can now use Hatcher's characterization of PL-fibration [H] to prove the following:

2.4. THEOREM. *If $\pi : K \to L$ is a PL-fibration with a compact fibre, then the pre-transfer map $\bar{\tau}_*$ defined above is a chain map.*

2.5. PROPOSITION. *Let $\pi : L \to K$ be a simplicial map. Assume that for each $x \in X$ the Euler characteristic $\chi(\pi^{-1}(x)) = k$, a fixed integer. Then the map $\Delta_q \times L \overset{\mathrm{id}\times\pi}{\to} \Delta_q \times K$ represents k-times the class of the identity map $\Delta_q \times K \to \Delta_q \times |K|$ in $\bar{C}_q(\Delta_q \times |K|)$.*

PROOF. It is enough to prove the result for $K = \Delta_n$ for $n \geqq 0$. In this case it is immediate from the equivalence relation, and the definition of $\chi(\pi^{-1}(X))$.

2.6. *Transfer for singular PL-fibrations.* We now show how to modify $\bar{\tau}_q$ for PL-fibration with a small singularity (compare [S, p. 491]) so as to be a transfer map τ_q on chains.

2.7. DEFINITION. A PL-fibration $\pi : K \to L$ with singularity along $W \subset L$ is a PL-map such that π is a Serre fibre map away from W.

EXAMPLE. Assume $G = \mathbf{Z}/p\mathbf{Z}$ acts simplicially on M, then $M \to M/G$ is a PL-fibration with singularity along $M^G \subset M/G$.

We say that the singularity $W \subset L$ in a singular PL-fibration is *small* if, for any point $w \in W$, there are arbitrarily small neighborhoods U of w such that $U - W$ is connected and non-empty: for example, if L is a manifold and W a subpolyhedron of codimension two.

Let $\pi : K \to L$ be a singular PL-fibre map with compact fibres, and singular subset $W \subset L$. Take a triangulation of L with W subcomplex. To define $\tau(\sigma_q)$ let $\sigma_q \in L$ be a q-simplex. If $\sigma_q \notin W$ we put $\tau(\sigma_q) = \bar{\tau}(\sigma_q)$. If $\sigma_q \in W$, it follows from the smallness condition on W that σ_q is a face of some $\sigma_{q+1} \in L - W$. Assume $\sigma_q = \sigma_{q+1}^{(i)}$ for some σ_{q+1} as above.

2.8. DEFINITION. Let $\sigma_q = \sigma_{q+1}^{(i)}$ for some σ_{q+1} as above. Set $\tau_q(\sigma_q) := \bar{\tau}^{(i)}(\sigma_{q+1})$, namely, the i-th face of the transfer on $\sigma_{q+1} \in L - W$.

2.9. PROPOSITION. *The value of $\tau_q(\sigma_q)$ in $\bar{C}_q(K)$ does not depend on the choice of the relation $\sigma_q < \sigma_{q+1}^{(i)}$.*

PROOF. Since W is small let U be a simplicial neighborhood of the barycenter $x \in \sigma_q$ such that $U - W$ is connected. We can assume that the star of σ_q is in U. Let $\bar{\sigma}_{q+1}$ be another simplex with $\bar{\sigma}_{q+1}^{(j)} = \sigma_q$, so that $\sigma_{q+1} = \langle V_i, \sigma_q \rangle$ while $\bar{\sigma}_{q+1} = \langle \bar{V}_j, \sigma_q \rangle$ and for some vertices $V_i, \bar{V}_j \in L - W$. By assumption of smallness we can assume that the one simplex $(V_i, \bar{V}_j)$ is in LW, since there must be a chain of 1-simplices connecting V_i to $\bar{V}_i$. Consider the $(q+2)$-simplex σ_{q+2} whose vertices are V_i, $\bar{V}_j$ and those of σ_q.

The map $\Delta_q \times F_i \to \pi^{-1}(\sigma_q)$ given by (4.19) is by definition the restriction of the canonical map

$$\tilde{\pi} : \Delta_{q+1} \times F_i \to \pi^{-1}(\sigma_i)$$

where F_i is $\pi^{-1}(\hat{\sigma}_i)$, the inverse image of the barycenter $\hat{\sigma}_i$ of σ_i. But since by [H] and Lemma (2.2) the map considered above

$$\psi : \Delta_{q+1} \times F_{\hat{\sigma}_{q+2}} \to \Delta_{q+1} \times F_i$$

gives the identity chain, one has that $\tau_q(\sigma_q)$ is also given by the restriction over $\sigma_q < \sigma_{q+2}$ of the canonical map

$$\Delta_{q+2} \times F_{\hat{\sigma}_{q+2}} \xrightarrow{\tilde{\pi}|} K$$

for the restriction

$$\tilde{\pi} \,\big|: \pi^{-1}(\sigma_{q+2}) \to \sigma_{q+2}.$$

By the same token this last chain is equal to the corresponding chain defined using $\bar{\sigma}_{q+1} < \sigma_{q+2}$.

In conclusion one has:

2.10. THEOREM. *For any PL-fibre map $K \to L$ with small singularity set $W \subset L$ there is a chain map*

$$\tau : C_*(L) \to CC_*(K)$$

where C_q denotes simplicial chains. This chain-level transfer has the usual properties:

(i) *The composition*

$$C_q(L) \to CC_q(K) \to CC_q(L)$$

induces multiplication by $\chi(F)$ on homology groups.

(ii) *τ is natural w.r.t. inclusion of subpolyhedra $L_1 \subset L$.*

2.11. REMARK. It can be easily seen that the transfer map for singular fibre maps cannot, in general, respect the Steenrod algebra action in (co)-homology. One considers the well-known map $CP^2 \to S^4$ gotten by dividing out the conjugation action on CP^2 by complex numbers z with $|z| = 1$. The singularity subset in $RP^2 \subseteq S^4$ over which the fibres consist on only one point.

PROOF. Condition (ii) follows from (2.2), while (i) is immediate from 2.5.

2.12. *Properties* (1)–(3). It is not difficult to verify the validity of naturality with respect to composition. Notice that (3) follows very naturally in the present context.

3. Oliver transfer

Assume G is a compact Lie group acting on the left on X. In many situations one may assume that $X \to X/G$, the projection to the orbit space [which is written here to the right to conform to standard notation], has nice simplicial triangulation in the sense of Illman [I]. In this case one can decompose X as a union of Illman's equivariant simplices

$$X = \coprod_{\sigma_n \in X/G} \Delta_n(G; H_0^\sigma, \ldots, H_n^\sigma) \tag{3.1}$$

where the union is taken over all simplices of X/G and $H_0^\sigma \subset \cdots \subset H_n^\sigma$ are closed subgroups of G. If $\tau_n^{(i)} < \tau_n$ is a face relation in X/G, then

$$\Delta_n(G; H_i^\sigma)_{i=0}^{n-1} < \Delta_n(G; H_i^\tau)_{i=0}^{n}$$

is also a face relation in X.

Now let H be a closed subgroup of G. We define a transfer chain map

$$\tau_* : C_*(X/G) \to CC_*(X/H)$$

from the *simplicial* chain complex of X/G using the above triangulation as follows: For every σ_n generator of $C_n(X/G)$ consider the inclusion

$$\Delta_n(G; H_0^\sigma \cdots H_n^\sigma) \to X$$

of [I]. This gives a map

$$\tilde{\tau}(\sigma) : \Delta_n \times G \to \Delta_n(G, H^\sigma) \to X.$$

By factoring out both sides by the action of H we obtain a map:

$$\tilde{\tau}/H : \Delta_n \times G/H \to X/H.$$

Now since G/H is a compact polyhedron we define

$$\tau_n(\sigma) = [\tilde{\tau}/H] \in C_n(X/H). \tag{3.2}$$

We need to show that τ_n commutes with face operators. But this is checked over each closed simplex. So we may assume $X/G = \Delta_n$. The commutativity now follows from that of the following canonical diagram:

$$\begin{tikzcd}
\Delta_{n-1} \times G \arrow[r] \arrow[d] & \Delta_n \times G \arrow[d] & \\
\Delta_{n-1}(G; H_i) \arrow[r] \arrow[d] & \Delta_n(G; H_i) \arrow[r] \arrow[d] & X \arrow[d] \\
\Delta_{n-1} \arrow[r, "d_i"'] & \Delta_n \arrow[r] & X/G
\end{tikzcd} \tag{3.3}$$

It is easy to see that τ_*, as a chain map, has all the usual properties of transfer.

3.4. *Reducing the dimension of the singular subset.* S. Weinberger suggested that the construction of Oliver's transfer for a map $X/H \to X/G$ can be gotten directly from 2.10 above by taking a product $X \times S^\rho$ so as to increase the codimension of the singularity set $X \subset X/G$. This can be achieved by picking a linear representation, say $\rho: G \to \mathbf{C}^n$ with $S^\rho = (\mathbf{C}^n)/\infty$ the corresponding sphere with an action of G that leaves the point $0, \infty \in S^\rho$ fixed. We choose the representation ρ such that $S^\rho/H \to S^\rho/G$ is a singular fibration with fibre G/H and with singularity set W of codimension $\geqq 2$. Now if W_ρ is the singularity set in S^ρ/G, then the orbit over $S^\rho/G - W$ is G/H thus the orbit over $(S^\rho/G) \times X - W \times X$ is also G/H and the singularity is at most $W \times X$, which is of codimension at least 2 in $(S^\rho/G) \times X$. Since $\{\infty\} \times X$ is a retract of $S^\rho \times X$ in a G-equivariant way the map $X/H \to X/G$ is a retract of $S^\rho \times_H X \to S^\rho \times_G H$; we get a transfer to the former as a retract of the transfer to the latter guaranteed by 2.10 above.

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